An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle

E. E. Fisher*
Honeywell Inc., St. Paul, Minn.

This paper is concerned with the application of optimal linear-control theory to the flexible-vehicle problem. As a matter of example, it develops a quadratic cost-criterion control law and seeks to approximate this required feedback by using a multiple-sensor scheme. The number of sensors required by the proposed scheme is less than the order of the plant dynamics.

Nomenclature

F = thrust force
X = axial air force
N' = air force perpendicular to long axis of vehicle
R' = control force perpendicular to long axis of vehicle
m = mass of vehicle
z = displacement of mass center of vehicle perpendicular to standard path
α = angle of attack
β = swivel motor deflection or vane deflection
φ = attitude angle
V_0 = velocity of vehicle along its path
η_i = normalized flexure modes
d_i = const
c_1 = const
c_2 = const
e_i = const
f_i = const
ζ_i = flexure-mode damping ratio
ω_i = flexure-mode frequency
$Y_i(y)$ = normalized flexure-mode shape as function of y, the distance from tail
y_β = engine pivot point
y_{cg} = position of mass center with respect to vehicle tail
y = parameter measuring distance from tail of vehicle

Introduction

THE major difficulty encountered in applying the optimal theories of control to the control of the flexible vehicle is the determination of state vector components. Optimal theories determine the control law as a feedback of state vector components. Sensors, however, measure a blend of state vector components. Elementary considerations seem to establish that as many sensors are needed as there are state vector components. This paper is concerned with approximating a feedback law given by the quadratic cost criterion with less sensors than state vector components.

Recall that the quadratic cost-criterion controller is concerned with the solution to the regulator problem (the problem of transferring a given initial state to the origin), which minimizes a certain quadratic cost functional for a system governed by a linear ordinary differential equation. It is shown[5] that the optimal control for autonomous systems with constant weight penalty and infinite penalty time is, in fact, a constant-gain linear feedback law. In what follows, calculations are performed to establish a particular quadratic criterion controller. A method of approximating this desired feedback law, using less sensors than state vector components, then is established.

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* Senior Research Scientist MPG Research Laboratory.

Vehicle Dynamics

The dynamical equations of a typical five-engine flexible rocket booster are presented. Consideration is given to rigid body and three flexure degrees of freedom. Fuel sloshing and engine dynamics are not considered.† Linear equations with constant coefficients are assumed. The motion considered is in the pitch plane. With recourse to the definitions, the equations of motion are

$$\ddot{z} = \frac{F - X}{m}\varphi + \frac{N'}{m}a + \frac{R'}{m}\beta + \sum_{j=1}^{3} \mathrm{d}_j\eta_j \qquad \text{lateral-path motion} \tag{1}$$

$$\ddot{\varphi} + c_1 a + c_2\beta + \sum_{j=1}^{3} e_i\eta_i = 0 \qquad \text{angular motion} \tag{2}$$

$$a = \varphi - (\dot{z}/v_0) \qquad \text{angular relationship} \tag{3}$$

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = f_i\beta \qquad \text{flexure motion} \qquad i = 1, 2, 3 \tag{4}$$

Here ($\dot{}$) represents differentiation with respect to time. New variables are defined so that the equations can be rewritten in the standard state vector form. The change of variables is defined in Eq. (5) as

$$\left.\begin{matrix} x_1 = \varphi & x_2 = \dot{\varphi} & x_3 = \alpha & x_4 = \eta_1 \\ x_5 = \dot{\eta}_1 & x_6 = \eta_2 & x_7 = \dot{\eta}_2 & x_8 = \eta_3 \\ & x_9 = \dot{\eta}_3 & \beta = u & \end{matrix}\right\} \tag{5}$$

In this transformation, the equations of motion become those given by

$$\dot{x} = Ax + bu \tag{6}$$

Here x is a column vector with components $x_1, x_2, \ldots,$ and x_9, and A and b are matrices (as listed) with a_{ij} and b_i as appropriate constants. Thus

$$A = \begin{vmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{22} & 0 & a_{28} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & a_{38} & 0 & a_{38} & 0 \\ 0 & 0 & 0 & 0 & a_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{76} & a_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{98} & a_{99} \end{vmatrix} \quad b = \begin{vmatrix} 0 \\ b_2 \\ b_3 \\ 0 \\ b_5 \\ 0 \\ b_7 \\ 0 \\ b_9 \end{vmatrix} \tag{7}$$

Of prime concern is the information available from sensors

† It is known that fuel sloshing can be neglected justifiably in calculations such as this; however, for an accurate physical model, engine dynamics probably should be retained.

for feedback purposes. If $s_a(y)$ is the output of an accelerometer located at a station y meters from the tail of the rocket booster, it can be shown that $s_a(y)$ will be given by (see Nomenclature)

$$s_a(y) = \frac{R'}{m}\left\{\beta + \sum_{i=1}^{3} Y_i(y_\beta)\eta_i\right\} + \frac{N'}{m}\alpha + (y_{cg} - y)\ddot{\varphi} + \sum_{i=1}^{3} Y_i(y_\beta)\ddot{\eta}_i - \sum_{i=1}^{3}\left[\frac{F}{m}Y_i(y_\beta) - \frac{(F-X)}{m}Y_i(y)\right]\eta_i \quad (8)$$

If $\ddot{\varphi}$ and $\ddot{\eta}_i$ are replaced in Eq. (8) by their values given in Eqs. (2) and (4), there results

$$s_a(y) = \sum_{j=3}^{9} z_{aj}(y)x_j + z_{a0}u \quad (9)$$

Here the coefficients $z_{aj}(y)$ are functions that depend upon y either linearly or through the mode slope functions $Y_i(y)$. It is noticed that, for a given fixed sensor location y, the output of an accelerometer is a linear nonhomogeneous function of the state variables x_1, x_2, . . . , and x_9. Since the nonhomogeneous term $z_{a0}u$ can be subtracted from the sensor output in principle, $s_a(y)$ henceforth will be considered as given by Eq. (9) with z_{a0} set equal to zero.

If $s_r(y)$ is the output of a rate gyro located y meters from the rocket booster's tail, it can be shown that

$$s_r(y) = \dot{\varphi} + \sum_{i=1}^{3} Y_i(y)\dot{\eta}_i \quad (10)$$

Finally, if $s_p(y)$ is the output of a position gyro (angular position) located y meters from the rocket booster's tail, it can be shown that

$$s_p(y) = \varphi + \sum_{i=1}^{3} Y_i(y)\dot{\eta}_i \quad (11)$$

The important thing to be noticed is that both rate and position gyros measure linear combinations of the state variables x_1, x_2, . . . , and x_9. To form a correspondence between the outputs $s_r(y)$, $s_p(y)$, and $s_a(y)$, Eqs. (10) and (11) will be rewritten as

$$\left.\begin{aligned} s_r(y) &= \sum_{j=2,5,7,9} z_{rj}(y)x_i \\ s_p(y) &= \sum_{j=1,4,6,8} z_{pj}(y)x_j \end{aligned}\right\} \quad (12)$$

Here the coefficients $z_{rj}(y)$ and $z_{pj}(y)$ are the appropriate functions of y. In what follows, any sensor in which the output is a linear homogeneous function of the state variables (as the preceding equations are) will be called a linear sensor.

If the output of nine linear sensors is given, and if the associated matrix (which depends upon the locations of the nine sensors, as well as their type) is nonsingular, then the nine time functions x_1, x_2, . . . , and x_9 can be expressed as a linear combination of the nine sensor outputs. Thus any control law (a rule that determines u as a function of x_1, . . . , and x_9 is called a control law) can be constructed, in principle. Later, consideration is given to the problem of implementing a given linear control law, using less than nine sensors. This suffices to introduce plant and sensor dynamics; thus attention is turned to controller design.

Design of Vehicle Controller by the Quadratic Criterion

The quadratic criterion is discussed with comments about its applicability to the flexible-vehicle problem. The design of a controller for the given vehicle then is presented.

Quadratic Penalty-Function Criterion

Let a system be governed by the state equation

$$\dot{x} = Ax + bu \quad (13)$$

where x is an n vector, which represents the state of the system, A is a constant n- by-n matrix, b is a constant n vector, and u is a scalar. Consider the problem of determining a control law

$$u = k'x \quad (14)$$

where k is an n-dimensional constant vector, and ( )′ indicates transpose. For a given positive-definite, symmetric matrix Q, a control law $u = k'x$ is desired which minimizes $V(t)$ given by

$$2V(t) = \lim_{T\to\infty} \int_t^T (x'Qx + u^2)d\tau \quad (15)$$

Here t is the present time. Kalman[3] has shown that, under appropriate conditions of controllability, there exists a unique, positive-definite matrix P, which satisfies the matrix equation

$$A'P + PA - Pbb'P + Q = 0 \quad (16)$$

such that the control $u = k'x$, with k given by

$$k = -Pb \quad (17)$$

is the unique optimum control.

Theoretically, the optimal control is optimal. In practice, more is required of the control than can be specified by a simple mathematical penalty. For instance, the exact physical characteristics required of a control are stabilization of the unstable air frame, small deviation from the desired attitude, small angle of attack to reduce bending loads, bounded engine deflection and rates, etc. With the exception of stability, none of these things results automatically from the quadratic criterion.

On the other hand it is believed that some proper (though empirically chosen) penalty matrix Q will produce a control that is a reasonable compromise of the preceding objectives. Since this "proper" matrix is not known, designing via the quadratic penalty criterion is an iterative process that starts with estimating the weighting matrix Q on the basis of physical considerations. Equation 16 is then solved for the associated P matrix. The gain vector k is established by Eq. (17), and the closed-loop system is examined for physical characteristics. This leads to the need for a readjustment of the weighting matrix Q, and the process is thus

Table 1 Dynamical constants for a typical flexible vehicle

$$A = \begin{bmatrix} 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2165 & -0.0356 & 0 & -0.0299 & 0 & -0.0270 & 0 \\ -0.0458 & 1.000 & -0.0133 & 0.0004 & 0 & 0.0006 & 0 & 0.0007 & 0 \\ 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -29.81 & -0.0546 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -169.0 & -0.1300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -334.3 & -0.1828 \end{bmatrix}$$

$$B' = [\,0 \quad -1.138 \quad -0.0348 \quad 0 \quad 29.56 \quad 0 \quad 47.25 \quad 0 \quad 16.40\,]$$

iterated. At any one step of this process, it is not a trivial matter to solve Eq. (16). For this reason an iterative procedure (to be explained later) is used.

Design of Controller for Given Vehicle

Numerical values for the matrices A and b are given in Table 1. The units used are meters, radians, and seconds. For purposes of comparison, the characteristic roots λ_1, λ_2, $\ldots$, and λ_9 of the matrix A are presented in Table 2.

High order in the differential equations of the plant, as well as a natural interest in rigid body motion, motivated a study of a fictitious rigid body [governed by Eqs. (18)]. A controller was designed for this rigid body by a series of adjustments and readjustment of the matrix Q^1. Thus

$$\left.\begin{aligned} &\dot{x}^1 = A^1x^1 + b^1u^1 \\ &x^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A^1 = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &b^1 = \begin{pmatrix} 0 \\ b_2 \\ b_3 \end{pmatrix} \qquad K^1 = \begin{pmatrix} k_1{}^1 \\ k_2{}^1 \\ k_3{}^1 \end{pmatrix} \\ &u^1 = k^1x^1 \end{aligned}\right\} \tag{18}$$

A suitable weighting matrix Q^1, with corresponding matrix P^1, was arrived at after several iterations. The resulting closed-loop characteristic roots were used as a criterion for the final selection of Q^1. That is, the desired weighting matrix Q^1 was chosen on the basis of the characteristic roots of the matrix $A^1 + b^1k^1$, which resulted from Q^1. Characteristic roots and associated gains are presented in Table 3. Weighting matrices used were diagonal and of the form $Q^1 = (\delta_{ij}\, q_i{}^1) = (q_{ij}{}^1)$, $q_i{}^1 > 0$. (This is the most simple positive, definite, symmetric matrix and provides a method of directly weighting each state component.) The controller listed as case 4 was considered as the desired rigid body controller.‡

A fifth-order system consisting of a fictitious rigid body with one flexure mode was considered next. Its dynamics are governed by

$$\left.\begin{aligned} &\dot{x}^2 - A^2x^2 + b^2u^2 \\ &x^2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad A^2 = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix} \\ &b^2 = \begin{pmatrix} 0 \\ b_2 \\ b_3 \\ 0 \\ b_5 \end{pmatrix} \qquad k^2 = \begin{pmatrix} k_1{}^2 \\ k_2{}^2 \\ k_3{}^2 \\ k_4{}^2 \\ k_5{}^2 \end{pmatrix} \end{aligned}\right\} \tag{19}$$

Here the weighting matrix $Q^2 = (\delta_{ij}q_i{}^2)$ was used, where the elements $q_1{}^2$, $q_2{}^2$, and $q_3{}^2$ were equal to $q_1{}^1$, $q_2{}^1$, and $q_3{}^1$, respectively; $q_5{}^2$ and $q_5{}^2$ were chosen quite small. This building process was motivated by difficulties in solving Eq. (16) for high-order systems. Although various procedures were tried in its solution, integration of the Riccati matrix differential

Table 2 Characteristic roots of the open-loop system

$\lambda_1 = 0.046$	$\lambda_2 = 0.4333$	$\lambda_3 = -0.493$
$\lambda_4 = -0.0270 + 5.460i$	$\lambda_5 = -0.0270 - 5.460i$	$\lambda_6 = -0.0650 + 1299i$
$\lambda_7 = -0.0650 - 12.99i$	$\lambda_8 = -0.0910 + 1.828i$	$\lambda_9 = -0.091 - 18.28i$

Table 3 Rigid body controllers for several choices of the weighting matrix Q^1

	$q_1{}^1$	$q_2{}^1$	$q_3{}^1$	$\lambda_1{}^1$	$\lambda_2{}^1$	$\lambda_3{}^1$	$k_1{}^1$	$k_2{}^1$	$k_3{}^1$
Case 1	2	2	2	−0.035	−1.378 +0.6230i	−1.378 −0.6230i	2.508	2.450	−0.2720
Case 2	0.5	0.5	0.5	−0.036	−0.922 +0.556i	−0.922 −0.556i	1.280	1.641	−0.0470
Case 3	0.1	1.0	0.1	−0.004	−0.500	−1.216	0.6290	1.497	0.0710
Case 4	0.1	0.05	0.5	−0.043	−0.761 +0.574i	−0.761 −0.5741i	1.096	1.365	−0.078

Table 4 Closed-loop characteristic roots and feedback gains for several weighting matrices applied to a ninth-order plant[a]

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
Case 1[b]	−0.0427	−0.7700 +0.588i	−0.770 −0.588i	−1.486 +5.28i	−1.486 −5.28i	−2.283 +12.77i	−2.283 −12.77i	−0.726 +18.25i	−0.726 −18.25i
Case 2[c]	−0.0427	−0.772 +0.588i	−0.772 −0.588i	−0.474 +5.44i	−0.474 −5.44i	−0.749 +12.98i	−0.749 −12.89i	−0.272 +18.28i	−0.272 −18.28i
Case 3[d]	−0.0427	−0.773 +0.587i	−0.773 −0.587i	−0.157 +5.46i	−0.157 −5.46i	−0.246 +13.00	−0.246 −13.00i	−0.121 +18.28i	−0.121 −18.28i

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	kl	k_9
Case 1[b]	1.099	1.464	−0.0515	−0.0625	−0.1007	−0.3450	−0.0938	−1.0134	−0.0682
Case 2[c]	1.099	1.374	−0.0785	−0.0375	−0.0311	−0.0662	−0.0288	−0.1162	−0.0216
Case 3[d]	1.099	1.345	−0.0865	−0.0145	−0.0100	−0.0143	−0.0078	−0.0094	−0.0037

[a] $q_1 = 0.1$, $q_2 = 0.05$, $q_3 = 0.5$, $q^c = W$, $j = 4, 5, \ldots$, and 9.
[b] Case 1: $W = 0.01$.
[c] Case 2: $W = 0.001$.
[d] Case 3: $W = 0.0001$.

‡ It has been indicated by the reviewer, though not checked by the author, that this method of iteration has been used in Air Force Documents ASD-TR-61-27 and ASD-TDR-63-376.

Table 5 Characteristic roots and gains for a series of problems of controller design with increasing plant order[a]

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
Case 1[b]	−0.0427	−0.760	−0.760						
		+0.574i	−0.574i						
Case 2[c]	−0.0427	−0.770	−0.770	−0.152					
		+0.585i	+5.46i	−5.46i					
Case 3[d]	−0.0427	−0.772	−0.772	−0.153	−0.153	−0.246	−0.246		
		+0.587i	−0.587i	+5.46i	+5.46i	+13.0i	−13.0i		
Case 4[e]	0.0427	−0.773	−0.773	0.157	−0.157	−0.246	−0.246	−0.121	−0.121
	−0.0427	+0.587i	−0.587i	+5.46i	+5.46i	+13.0i	−13.0i	+18.28i	−18.28i

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
Case 1[b]	1.096	1.465	−0.0778						
Case 2[c]	1.096	1.346	−0.0842	−0.0145	−0.0097				
Case 3[d]	1.099	1.345	−0.0860	−0.0140	−0.0097	−0.0097	−0.0078		
Case 4[e]	1.099	1.345	−0.0865	−0.0145	−0.0100	−0.0143	−0.0078	−0.0094	−0.0037

[a] $q_1 = 0.1$, $q_2 = 0.05$, $q_3 = 0.5$, and $q_j = 0.001 j > 3$.
[b] Case 1: rigid body third-order plant.
[c] Case 2: rigid body +1 flexure mode fifth-order plant.
[d] Case 3: rigid body +2 flexure modes seventh-order plant.
[e] Case 4: rigid body +3 flexure modes ninth-order plant.

equation to a steady state finally was adopted as a method of solution.§ Thus

$$-dp/dt = AP + AP - Pbb\,P + Q \tag{20}$$

An initial estimate of P^2 [the solution of Eq. (16) for A^2, B^2, and Q^2] was a matrix with its fourth and fifth rows and columns set equal to zero and its remaining nine elements set equal to the corresponding ones in P^1 [the solution of Eq. (16) for A^1, B^1, and Q^1]. This was used as an initial condition for the Riccati equation, which converged from there quite rapidly. The process was extended to seventh- and finally ninth-order systems for a variety of cases. Some of the results for ninth-order systems are presented in Table 4. In each case, it was found that the characteristic roots and associated gains changed very little as the system was built up from the rigid body problem. Table 5 indicates the progress of such a process for the characteristic roots and associated gains. Table 6 indicates the progress for the P matrices. Case 1 corresponds to a rigid body problem with weighting $q_1 = 0.1$, $q_2 = 0.05$, and $q_3 = 0.5$. Each successive case had two more variables added (i.e., one more flexure mode) with additional weighting equal to 0.0001.

It is noted that the weighting factor w, defined in Table 5, has little effect on the closed-loop flexure frequencies and that its effect on closed-loop damping goes as the square root of w. Another trend as pointed out by Reynolds and Rynaski[1] is that the coefficients of the characteristic equation increase in magnitude with an increase in the norm of the matrix Q. Previous experience with the flexible-vehicle problem indicates that the controller listed as case 4 in Tables 5 and 6 is a reasonable controller. Consideration will be given next to its implementation.

Construction of Linear Control Law Given by Quadratic Criterion

A general linear sensor with location y_i is defined to be such that its output $s_i(y_i)$ is given by

$$s_i(y_i) = \sum_{j=1}^{9} z_{ij}(y_i)x_j \tag{21}$$

Here the coefficients z_{ij} depend in some fashion upon y_i as the equations indicate. It is noted that s_i and x_i are time

Table 6 P matrices for a series of problems of controller design with increasing plant order[a]

Case 1	13.492	1.3302	−12.016						
		1.2120	−0.4243						
			11.641						
Case 2	13.524	1.3020	−12.063	−0.01074	−0.00119				
		1.1703	−0.41726	−0.0125	−0.00106				
			11.691	0.00111	0.00054				
				0.00869	0.00029				
Case 3	13.541	1.2956	−12.082	−0.01075	−0.00120	−0.008597	−0.000178		
		1.1610	−0.41553	−0.012507	−0.001083	−0.010426	−0.000138		
			11.710	0.001108	0.000535	0.000725	0.000096		
				0.00872	0.000013	0.000100	−0.000012		
					0.000289	0.000088	−0.000000		
						0.027456	0.000001		
							0.000162		
Case 4	13.548	1.2951	−12.089	−0.011071	−0.001196	−0.008633	−0.000180	−0.004266	−0.000098
		1.1598	−0.41554	−0.012897	−0.001078	−0.010476	−0.000180	−0.004266	−0.000068
			11.716	0.001135	0.000531	0.000722	0.000095	0.000542	0.000052
				0.009015	0.000017	0.000105	−0.000013	0.000049	−0.000002
					0.000299	0.000092	0.000000	0.000039	−0.000000
						0.027603	0.000002	0.010012	−0.000028
							0.000162	0.000056	−0.000000
								0.073434	−0.000000
									0.000220

[a] All matrices are symmetric.

§ It is easily seen that, if $P(t)$ is a matrix that satisfies Eq. (20) and has reached a steady state equal to P, a constant, then P satisfies Eq. (16). Kalman[4] shows that such a steady state is indeed the unique positive, definite, solution of Eq. (16).

Table 7 Sensor gains and resulting feedback using six sensors

	i	1	2	3	4	5	6	7	8	9
Desired feedback	k^i	1.099	1.345	−0.0865	−0.0145	−0.0100	−0.0143	−0.0078	−0.0094	−0.0037
Case 1	y^i	50	25	25	50	50	25			
	γ^i	−0.0028	0.0028	1.119	0.2248	0.2570				
	k^i	1.098	1.344	−0.0860	−0.0348	−0.0386	−0.0049	0.0079	−0.0123	0.0172
Case 2	y^i	50	25	25	75	50	25			
	γ^i	−0.0028	−0.0028	1.176	0.1685	0.2570	0.8413			
	k^i	1.09	1.345	−0.086	−0.040	−0.0256	−0.0025	−0.0125	−0.0131	−0.0135
Case 3	y^i	50	25	25	75	75	25			
	γ^i	−0.0030	+0.0030	1.760	0.1685	0.2157	0.8832			
	k^i	1.099	1.345	−0.086	−0.0190	−0.0256	−0.006	−0.0256	−0.0131	−0.0135

functions, although this dependence is not indicated. Let m sensors of the previous nature be stationed at locations y_i, y_2, . . . , and y_m where $m \leq 9$. Let the output of each sensor be multiplied by an as yet undetermined sensor gain γ_i and summed to form a feedback quantity $\tilde{u}$ given by

$$\tilde{u} = \sum_{i=1}^{m} \gamma_i s_i = \sum_{i=1}^{m} \gamma_i \left(\sum_{j=1}^{9} z_{ij} x_j \right) \tag{22}$$

If k_1, k_2, . . . , and k_9 are the feedback gains given by the quadratic criterion (k_1, k_2, . . . , and k_9 are the components of the vector k as previously defined), the desired feedback u is given by

$$u = \sum_{j=1}^{6} k_j x_j \tag{23}$$

The quantity u will be equal to u if

$$\sum_{i=1}^{m} \gamma_i z_{ij} = k_j \qquad j = 1, 2, \ldots, 9 \tag{24}$$

This equation can be interpreted as a vector equation in the vectors z_i and k, which are given by

$$\begin{aligned} z_i &= (z_{i1}, z_{i2}, \ldots, z_{i9})' \\ k &= (k_1, k_2, \ldots, k_9)' \end{aligned} \tag{25}$$

The vector equation is then that given by

$$\sum_{i=1}^{m} \gamma_i z_i = k \tag{26}$$

In the event that m is 9 and the vectors z_i are linearly independent (i.e., they span the nine-dimensional space), there exist constants γ_i such that Eq. (26) is true for any given vector k. This is a statement of the fact that nine independent¶ sensors serve to determine the nine components of the state vector. (This problem is discussed in detail by Harvey.[2])

If m is less than nine, Eq. (26) will not be satisfied for arbitrary k but only for those k, which lie in the space spanned by z_1, z_2, . . . , and z_m. Since k is given by the quadratic criterion and thus is fixed, solution of Eq. (26) with m less than nine involves selecting the z_i in such a way that they span a space, which includes the vector k. Because the vectors z_i each depend upon the position of the ith sensor y_i, there is some hope that, with proper sensor positioning, Eq. (26) can be solved for m less than nine.** Because of the physical nature of the problem, exact solutions are not necessary. For instance, the vector

$$\sum_{i=1}^{m}{}^{2} \gamma_i z_i$$

can be considered close to k ift he magnitude of their difference vector or its square is small. Its square is introduced therefore as an error quantity E given by

$$E = \|\Sigma \gamma_i z_i - k\|^2 \tag{27}$$

Consideration of the actual problem at hand dictates how the error function E is to be used. It is noted that the vectors $z_i(y_i)$ originate from accelerometers, rate gyros, or position gyros. That is, the m sensors indicated in Eq. (27) are as yet an undetermined combination of accelerometers, rate gyros, and position gyros. They are located at certain specific, though as yet undetermined, locations y_1, y_2, . . . , and y_m. As is usually the case, the error function E should be minimized with respect to its parameters to yield the best set of parameters. These parameters include the m sensor gains γ_i and the m sensor positions y_i. If E is minimized with respect to the sensor positions y_i, the mathematical minimum may result in certain of the y_i's being at the ends of the booster. This could happen because the variables y_i are defined only for $0 < y_i < l$ where l is the booster length. Also minimization with respect to the y_i's is not a linear problem since the y_i's enter the problem nonlinearly through the mode slope functions. For these reasons, the error function E will not be minimized with respect to the y_i's. Instead the positions y_i will be assumed given, and the error function E will be minimized (for the given y_i's) in terms of the γ_i's. The γ_i's, which minimize E for given fixed y_i's, are given by

Table 8 Closed-loop characteristic roots for the approximate feedbacks listed in Table 6

	i	1	2	3	4	5	6	7	8	9
Case 1	γ^i	−0.043	−0.769 +0.601i	−0.769 −0.601i	−0.572 −5.36i	−0.572 +5.36i	0.119 +13.0i	0.119 −13.0i	0.049 −18.3i	0.049 +18.3i
Case 2	γ^i	−0.043	−0.765 +0.598i	−0.765 −0.598i	−0.401 −5.45i	−0.401 +5.45i	−0.352 +12.9i	−0.352 −12.9i	−0.199 −18.3i	−0.199 +18.3i
Case 3	γ^i	−0.043	−0.782 +0.596i	−0.782 −0.596i	−0.386 −5.39i	−0.387 +5.39i	−0.352 +12.9i	−0.352 −12.9i	−0.199 +18.3i	−0.199 −18.3i

¶ For linear sensors, a set of sensors will be said to be independent if the associated vectors z_i are linearly independent. It is assumed tacitly that this can be achieved by the correct variety of sensors at the correct sensor locations.

** Any physically usable calculation, at this point, will have to consider the change in mode shapes. It is believed that some positions that work well on the average can be obtained, since the change in mode shapes is not too great. For this reason this will not be dealt with here.

the familiar, linear least-squares normal equations as

$$\sum_{j=1}^{m} \gamma_j (z_i \cdot z_j) = z_i \cdot k \qquad i = 1, 2, \ldots, m \tag{28}$$

[Here $(\cdot)$ indicates the vector dot product.] It is noted that, for many rocket booster problems, it may be advantageous to minimize E with respect to both the y_i's and γ_i's. This is not done here.

Attention is now turned to the calculation at hand, the approximation of the feedback derived. (This is displayed in Tables 5 and 6 as case 4.) Three basic types of sensors (accelerometers, rate gyros, and position gyros) are assumed. These will have sensor gain vectors z_a, z_r, and z_p given by the next equation [Eq. (29)]. The mode shape functions $Y_1(y)$, $Y_2(y)$, and $Y_3(y)$, in which the derivatives appear in Eq. (29), are given by the polynomials in Eq. (30). The location variable y is assumed to be in the interval (0,100):

$$\left.\begin{array}{l}
z_{a1}(y) = 0 \qquad z_{r1}(y) = 0 \qquad z_{p1}(y) = 1 \\
z_{a2}(y) = 0 \qquad z_{r2}(y) = 1 \qquad z_{p2}(y) = 0 \\
z_{a3}(y) = 39.28 - 1.138y \qquad z_{r3}(y) = 0 \qquad z_{p3}(y) = 0 \\
z_{a4}(y) = -20.89\ Y_1(y) - 0.036y - 29.02 \\
z_{r4}(y) = 0 \qquad z_{p4}(y) = Y_1(y) \\
z_{a5}(y) = -0.055 \qquad z_{r5}(y) = Y_1(y) \qquad z_{p5}(y) = 0 \\
z_{a6}(y) = -20.89\ Y_2(y) - 0.030y - 1685 \\
z_{r6}(y) = 0 \qquad z_{p6}(y) = Y_2(y) \\
z_{a7}(y) = -0.130 \qquad z_{r7}(y) = Y_2(y) \qquad z_{p7}(y) = 0 \\
z_{a8} = -20.89\ Y_3(y) - 0.027y - 333.9 \\
z_{r8}(y) = 0 \qquad z_{p8}(y) = Y_3(y) \\
z_{a9} = -0.183 \qquad z_{r9}(y) = Y_3(y) \qquad z_{p9}(y) = 0
\end{array}\right\} \tag{29}$$

$$\left.\begin{array}{l}
Y_1(y) = 0.1045 \times 10 - 0.6319 \times 10^{-1}y + 0.5917 \times \\
\quad 10^{-3}y^2 + 0.3273 \times 10^{-1}y^3 - 0.2429 \times 10^{-6}y^4 + \\
\qquad\qquad 0.4599\ 10^{-8} - 0.2156 \times 10^{-10}y \\
Y_2(y) = 0.1349 \times 10 - 0.7790 \times 10^{-1} - 0.5233 \times \\
\quad 10^{-2}y^2 + 0.4182 \times 10^{-3} - 0.1115 \times 10^{-4}y^4 + \\
0.1575\ 10^{-6}y^5 - 0.1167 \times 10^{-8}y^6 + 0.3451 \times 10^{-4}y^7 \\
Y_3(y) = 0.1079 \times 10 - 0.5980 \times 10^{-1} - 0.1566 \times \\
\quad 10^{-2}y^2 - 0.1956 \times 10^{-3}y^3 + 0.1931 \times 10^{-4}y^4 - \\
0.4760\ 10^{-6}y^5 + 0.4577 \times 10^{-8}y^6 - 0.1527 \times 10^{-10}y^7
\end{array}\right\} \tag{30}$$

Several sensors must be positioned in such a way that the gain vector k can be approximated by a linear combination of sensor outputs. An examination of the sensor gains [Eq. (29)] indicates that at least one accelerometer must be used in order to pick up x_3, the angle of attack; at least one rate gyro must be used to determine pitch rate x_2; and at least one position gyro must be used in order to determine pitch angle x_1. It was found that using two of each type of sensor resulted in approximate feedbacks quite close to the one desired. Three sensor stations were chosen at 25, 50, and 75 m, and various combinations of sensor locations were selected. That is, the six sensors were permuted between these three stations in a variety of ways. Not all of the least-squares solutions turned out to be good, or in fact stable, although, with six sensors, good solutions resulted in two-thirds of the cases. Three typical cases are presented in Table 7. Here y_1 and y_2 are accelerometer locations, y_3 and y_4 rate gyro locations, and y_5 and y_6 position gyro locations. The resulting gains $k_1, k_2, \ldots,$ and k_9 are, of course, components of the vector $\gamma_1 z_1 + \gamma_2 z_2 + \ldots + \gamma_9 z_9$. The desired gain vector k_1, $k_2, \ldots,$ and k_9, as listed in case 4 of Table 5, is presented in Table 7 for comparison. The resulting closed-loop characteristic roots are presented in Table 8. It is noted that in two cases the closed loop is somewhat tighter than desired, whereas in the other case the closed loop is not stable. The calculations performed tacitly assume that the controller is capable of responding to the signal $k_1x_1 + \ldots + k_9x_9$. Thus, it may be that too tight a controller results in gimbal angle saturation. This question was not investigated. Neither was the question of gust response. Finally it is stated that the sensor stations were selected completely arbitrarily, and several other selections of stations worked equally as well.

Conclusions

A linear controller for a typical rocket booster was designed by use of the quadratic penalty-function criterion, and its approximate implementation was accomplished, using less sensors than the plant order. It is believed that the procedures involved constitute the beginnings of a design method, which would be capable of better realization of optimal linear controllers for flexible vehicles. Such techniques combined with the usual techniques of filtering very high frequency dynamics out of the sensor signals could yield a new attack to the synthesis problem for control of a very flexible vehicle.

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